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Random Walks with Short Memory

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A restricted random walk on a d-dimensional cubic lattice with different probabilities for forward, backward, and sideward steps is studied. The analytic solution for the generating function, exact expressions for the second and fourth moments of displacements, and diffusion and Burnett coefficients are given, as well as a systematic asymptotic expansion for the probability distribution of long walks.

KEY WORDS: Restricted/correlated random walks; forward/backward jump models; random walks with persistence.

1. INTRODUCTION

There exist many random walk (RW) models in which the next step is influenced by one or a few previous steps. Such correlated RWs are Markov processes described by the Chapman–Kolmogorov or master equation[1] for the probability density that depends on the presently as well as on the previously occupied sites. For a recent review the reader is referred to Haus and Kehr.[2]

Such models are, in general, of interest in connection with the statistics of long-chain molecules or with transport in crystals, and in particular in connection with lattice models or cellular automata for Lorentz gases.[3] In the latter case a random fraction of lattice sites is filled with scatterers; the walker moves along straight lines and is deflected at scattering sites according to deterministic or stochastic scattering rules. The correlated RW models often provide reference models in the limits of high and/or low concentration of scatterers.

The models of interest in this paper are nearest neighbor hopping

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This paper is dedicated to Nico van Kampen.

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models on hypercubic lattices with memory over two successive steps only; namely, the RW has probabilities α, β, and γ for forward, backward, and sideward steps, respectively. The model with reduced reversals or backward jump model ($\beta \neq \alpha = \gamma$) was solved long ago by Domb and Fisher.[4] Reference 3 seems to suggest that in more general cases, where the master equation for the probability distribution is a system of $2d$ coupled linear equations, one needs to rely on numerical, rather than analytical, methods.

Recently, however, Claes and van der Broeck[5] solved the forward jump model ($\alpha \neq \beta = \gamma$). They observe that the structure of the master equation for their model is analogous to the BGK model in kinetic theory. The relationship with kinetic theory is even closer in the limit of large coordination numbers, where the transition kernel in the master equation approaches in fact the Lorentz–Boltzmann collision operator for the three-dimensional Lorentz gas. The solution of ref. 5 for the Fourier–Laplace transform of the probability distribution in this limiting case was derived by Hauge[6] using the same method.

The purpose of the present paper is to solve the general case of an RW with correlated nearest neighbor jumps on a hypercubic lattice by combining the methods of refs. 4 and 5. For this case, exact results have already been published for the mean square displacement.[7]

The plan of the paper is as follows: in Section 2, I calculate the generating function for the probability distribution from the master equation; exact results for the distribution are given in Section 3; moments and transport coefficients are calculated in Section 4; Section 5 uses a saddle point method to develop an asymptotic series for the probability distribution; and Section 6 deals with eigenvalues and eigenmodes of the master equation. I conclude with some additional remarks.

2. CHAPMAN–KOLMOGOROV EQUATION

Let $P_\nu(n, t)$ be the probability that the RW arrives at site n on the tth step ($t = 0, 1, 2,...$), coming from site $n - \nu$ or, equivalently, having a "velocity" ν at arrival. Here $n = (n_x, n_y,..., n_d)$ is a site of a d-dimensional cubic lattice having unit lattice distance and N sites in total; labels ν, μ refer to the 2D nearest neighbor lattice vectors: $+\hat{e}_x$, $-\hat{e}_x$, $+\hat{e}_y$, $-\hat{e}$,..., $+\hat{e}_y$, $-\hat{e}_y$,..., $+\hat{e}_d$, $-\hat{e}_d$. A summation over all previous history of the RW gives the total probability for a displacement n in t time steps, $P(n, t) = \sum_\nu P_\nu(n, t)$.

The conditional probability satisfies the master or Chapman–Kolmogorov equation

$$P_\nu(n, t+1) = \alpha P_\nu(n - \nu, t) + \beta P_{-\nu}(n - \nu, t) + \gamma \sum_{\nu' \neq \pm\nu} P_{\nu'}(n - \nu, t) \quad (1)$$

with initial condition $P_v(n, 0) = \delta_{n0}/2d$, since all incident velocities are equally probable. The probabilities for a forward step, $\alpha \equiv \gamma + \varepsilon$, a backward step, $\beta \equiv \gamma - \delta$, and a sideward step, γ, are normalized as

$$\alpha + \beta + 2(d-1)\gamma = 1 \qquad \text{or} \qquad 2d\gamma = 1 + \delta - \varepsilon \tag{2}$$

I define the corresponding generating functions $G(q, s)$ of $P(n, t)$ and its Mellin transform $F(q, t)$ as

$$G(q, s) = \sum_{t=0}^{\infty} s^t F(q, t) = \sum_{t=0}^{\infty} s^t \sum_n e^{-inq} P(n, t) \tag{3}$$

and similarly for $G_v(q, s)$ and $F_v(q, t)$ in terms of $P_v(n, t)$, where $nq = \sum_{\alpha=x}^{d} n_\alpha q_\alpha$ is a scalar product. Then Eq. (1) becomes

$$G_v - 1/2d = \varepsilon s e^{-iqv} G_v - \delta s e^{-iqv} G_{-v} + \gamma s e^{-iqv} G \tag{4}$$

For the unrestricted RW ($\varepsilon = \delta = 0$), summing this equation over v directly yields the standard result for the generating function $G = (1 - \phi s)^{-1}$ where $\phi(q)$ is the generating function for the first step:

$$\phi(q) = (1/2d) \sum_v e^{-iqv} = (1/d) \sum_{\alpha=x}^{d} \cos q_\alpha$$

In Claes and Van den Broeck's method for the forward jump model ($\delta = 0$), Eq. (4) is solved for G_v in terms of G. Subsequent summation over v yields a closed form for $G(q, s)$. Domb and Fisher's method for solving the backward jump model ($\varepsilon = 0$) is equivalent to combining Eq. (4) for v and $-v$, eliminating G_{-v}, which yields G_v in terms of G, and proceeding as above. In the general case, I follow the same procedure and write Eq. (4) as

$$\begin{pmatrix} 1 - \varepsilon s e^{-iqv} & \delta s e^{-iqv} \\ \delta s e^{iqv} & 1 - \varepsilon s e^{iqv} \end{pmatrix} \begin{pmatrix} G_v \\ G_{-v} \end{pmatrix} = \begin{pmatrix} 1/2d + \gamma s e^{-iqv} G \\ 1/2d + \gamma s e^{iqv} G \end{pmatrix} \tag{5}$$

Solving for G_v and summing for v yields the following result for the generating function:

$$\begin{aligned} G(q, s) &= \Gamma/\Delta \\ \Gamma &= (1/d) \sum_{\alpha=x}^{d} (1 - asc_\alpha)/[1 - (a+b)\, sc_\alpha + abs^2] \\ \Delta &= (1/d) \sum_{\alpha=x}^{d} [1 - (1+a)\, sc_\alpha + as^2]/[1 - (a+b)\, sc_\alpha + abs^2] \end{aligned} \tag{6}$$

where $c_\alpha = \cos q_\alpha$ ($\alpha = x, y, ..., d$) and

$$a = \alpha - \beta = \varepsilon + \delta, \qquad b = \alpha + \beta - 2\gamma = \varepsilon - \delta \tag{7}$$

For the backward jump model ($a = -b = \delta$), the generating function reduces to the result of ref. 4:

$$G(q, s) = (1 - \delta s\phi)/[1 - (1 + \delta)\, s\phi + \delta s^2] \tag{8}$$

For the forward jump model ($a = b = \varepsilon$), Eq. (6) can also be written in the somewhat simpler form of ref. 5:

$$G(q, s) = \frac{\sum_v 1/(1 - \varepsilon s e^{-iqv})}{\sum_v (1 - s e^{-iqv})/(1 - \varepsilon s e^{-iqv})} \tag{9}$$

where the v sum runs over the nearest neighbor vectors.

3. EXPANSIONS OF THE GENERATING FUNCTIONS

To obtain $P(n, t)$, one has to extract the Mellin transform $F(q, t)$ as the coefficient of s^t in the s expansion of $G(q, s)$ [see Eq. (3)] and invert the Fourier transform. Domb and Fisher have obtained expansions of $F(q, t)$ as finite polynomials in $\phi(q)$, that is, in the generating functions for unrestricted walks [see their Eq. (40)]. An analogous result can be obtained for the *forward jump model* by expanding Eq. (9) in powers of s, with the result

$$F(q, t) = \varepsilon^{t-1} \sum_{m=1}^{t} (\varepsilon^{-1} - 1)^{m-1} \sum_{\{l_i\}}{}' \prod_{i=1}^{m} \phi_{l_i}$$

with

$$\phi_l(q) = (1/2d) \sum_v e^{-iqvl} = (1/d) \sum_{\alpha = x}^{d} \cos lq_\alpha$$

The prime on the summation sign indicates that the sum runs over partitions $\{l_i\}$ with $l_1 + l_2 + \cdots + l_m = t$ and $l_i \geqslant 1$. For the general case (6), I have not found any tractable form.

4. MOMENTS AND TRANSPORT COEFFICIENTS

In order to compute the mean square end-to-end distance of a t-step restricted walk and other moments of the distribution, I expand the rhs of Eq. (3) in powers of q_α ($\alpha = x, y, ..., d$) to obtain

$$
\begin{aligned}
G(q, s) &= (1-s)^{-1} - (1/2) \sum_{\alpha\beta} q_\alpha q_\beta \langle n_\alpha n_\beta \rangle(s) \\
&\quad + (1/24) \sum_{\alpha\beta\gamma\delta} q_\alpha q_\beta q_\gamma q_\delta \langle n_\alpha n_\beta n_\gamma n_\delta \rangle(s) + \cdots \\
&= (1-s)^{-1} - (1/2)\langle n_x^2 \rangle(s) \sum_\alpha q_\alpha^2 + (1/24)\langle n_x^4 \rangle(s) \sum_\alpha q_\alpha^4 \\
&\quad + (3/24)\langle n_x^2 n_y^2 \rangle \sum_{\alpha \neq \beta}\sum q_\alpha^2 q_\beta^2 + \cdots
\end{aligned} \tag{10}
$$

Here $\langle n_\alpha n_\beta \cdots \rangle(s)$ is the Mellin transform of the moment of displacement $\langle n_\alpha n_\beta \cdots \rangle_t$ after t steps. The simplification in the last line is a consequence of the cubic symmetry, which implies that there exists only one *independent* second moment $\langle n_\alpha n_\beta \rangle = \langle n_x^2 \rangle \delta_{\alpha\beta}$ and only two fourth moments, $\langle n_x^4 \rangle$ and $\langle n_x^2 n_y^2 \rangle$. After performing a q expansion of $G(q, s)$ in Eq. (6), I find with the help of (10) for the Mellin transforms of the moments

$$
\begin{aligned}
\langle n_x^2 \rangle(s) &= (1/d)\, s(1+as)(1-s)^{-2}\,(1-as)^{-1} \\
\langle n_x^2 n_y^2 \rangle(s) &= (2/d^2)(1-b)\, s^2(1+as)^2\,(1-s)^{-3}\,(1-as)^{-2}\,(1-bs)^{-1} \\
\langle n_x^4 \rangle(s) &= \langle n_x^2 \rangle(s) + 3\langle n_x^2 n_y^2 \rangle(s) \\
&\quad + (6/d)(a+b)\, s^2(1+as)(1-s)^{-2}\,(1-as)^{-2}\,(1-bs)^{-1}
\end{aligned} \tag{11}
$$

The inverse Mellin transform is given by the following integral in the complex s plane along a closed counterclockwise contour C_0 around $s=0$:

$$
\langle n_\alpha n_\beta \cdots \rangle_t = \oint_{C_0} \frac{ds}{2\pi i} s^{-t-1} \langle n_\alpha n_\beta \cdots \rangle(s) \tag{12}
$$

As the integrands vanish sufficiently fast for $(s) \to \infty$, the integrals can be replaced by the sum of integrals around the closed clockwisecontours C_1, C_a, and C_b around the multiple poles of the integrand, located $s=1$, $s=1/a$, and $s=1/b$, respectively. After lengthy but straightforward calculations, I find

$$
\langle n_x^2 \rangle_t = \frac{1}{d}\left[\frac{1+a}{1-a}t - \frac{2a(1-a^t)}{1-a^2}\right] \tag{13a}
$$

$$
\begin{aligned}
\langle n_x^2 n_y^2 \rangle_t &= \frac{1}{d^2}\left(\frac{1+a}{1-a}\right)^2 \left[t^2 - t\left(\frac{8a}{1-a^2} + \frac{1+b}{1-b}\right)\right. \\
&\quad \left. + \frac{8a(1+2a)}{(1-a^2)^2} + \frac{8ab}{(1-a^2)(1-b)} + \frac{2b}{(1-b)^2}\right] \\
&\quad - \frac{8(1-b)\,a^{t+2}}{d^2(1-a)^3\,(a-b)}\left(t + \frac{1+2a}{1-a} - \frac{b}{a-b}\right) - \frac{2}{d^2}\frac{(a+b)^2\,b^{t+1}}{(1-b)^2\,(a-b)^2}
\end{aligned} \tag{13b}
$$

$$\begin{aligned}
&\langle n_x^4\rangle_t - 3\langle n_x^2 n_y^2\rangle_t - \langle n_x^2\rangle_t \\
&\quad = \frac{6(a+b)(1+a)}{d(1-a)^2\,(a-b)}\left[t - \frac{a(3+a)}{1-a^2} - \frac{1}{1-b}\right] + \frac{6(a+b)^2\, b^{t+1}}{d(a-b)^2\,(1-b)^2} \\
&\quad\quad + \frac{12(a+b)\, a^{t+1}}{d(1-a)^2\,(a-b)}\left[t + \frac{3+a}{2(1-a)} - \frac{a}{a-b}\right]
\end{aligned} \tag{13c}$$

The pole at $s=1$ yields contributions to the moments that are linear or quadratic polynomials in t; the poles at $1/a$ and $1/b$ yield exponentially decaying terms a^t and b^t, where Eqs. (2) and (7) imply that $|a|$ and $|b|$ are smaller than unity. In fact, if I combine the probabilities α, β, and γ for a transition from "velocity" μ to "velocity" ν into a $2d$-dimensional matrix $W_{\nu\mu}$, it has only three different eigenvalues due to cubic symmetry, namely: 1 (nondegenerate), a (d-fold degenerate), and b [$(d-1)$-fold degenerate].

The result (13a) for the second moment holds for all models considered and has been derived in refs. 3–6. Higher moments, viz. $\langle n^4\rangle_t$ and $\langle n^6\rangle_t$, have only been calculated for the forward jump model,[5] which has the accidental degeneracy $a=b$. To obtain similar results for this model from (13b) and (13c), I cannot simply set $a=b$, because of factors $(a-b)^{-2}$ and $(a-b)^{-1}$. However, if one sets $b=a(1+\eta)$ and expands the coefficients in (13b) and (13c) in powers of η, then the singular terms proportional to η^{-2} and η^{-1} cancel. To obtain explicit expressions for the forward jump model, it is easier to set $a=b$ in Eq. (11) and invert the Mellin transforms directly.

As the displacement or end-to-end distance n after a large number of steps t is a Gaussian random variable, its probability distribution approaches a Gaussian of which all cumulants vanish except the second. The long-time behavior of the cumulants is therefore a measure of the approach to the limiting distribution. It is found that the $2m$th moment $\langle n_x^{2m}\rangle_t$ increases as t^m at large t, whereas the corresponding cumulant $\langle\langle n_x^{2m}\rangle\rangle_t$ only increases linearly with t. The coefficients of these linear terms define the diffusion coefficient D, Burnett coefficients B_{xx} and B_{xy}, super-Burnett coefficients, etc., in the following manner:

$$(1/2)\langle n_x^2\rangle_t \simeq Dt$$

$$(3/24)(\langle n_x^2 n_y^2\rangle_t - \langle n_x^2\rangle_t^2) \simeq B_{xy}\, t$$

$$(1/24)(\langle n_x^4\rangle_t - 3\langle n_x^2\rangle_t^2) \simeq B_{xx}\, t$$

Because of the cubic symmetry there are two different Burnett coefficients; in case of isotropic symmetry there is only one. The values of these transport coefficients here are

$$\begin{aligned} D &= (1/2d)(1+a)/(1-a) \\ B_{xy} &= -(1/2)\, D^2[4a/(1-a^2)+(1+b)/(1-b)] \\ B_{xx} &= B_{xy}+(1/12)\, D\{1+6(a+b)/[(1-a)(1-b)]\} \end{aligned} \tag{14}$$

5. ASYMPTOTICS FOR THE PROBABILITY DISTRIBUTION

As already mentioned in Section 4, the probability distribution $P(n, t)$ of the end-to-end distance approaches a Gaussian at large t. I want to study this approach.

For long walks the typical end-to-end distance $|n|$ increases like $\sqrt{t}$. In the coupled "diffusive" limit with $|n|$ and t large and $|n|^2/t$ fixed, one can obtain the asymptotic behavior of $P(n, t)$ as a systematic expansion in powers of $1/t$. The starting point is the inversion of Eq. (3):

$$P(n, t)=(2\pi)^{-d}\int_{-\pi}^{\pi}\cdots\int d^{(d)}q\, e^{-inq}(2\pi i)^{-1}\oint_{C_0} ds\, s^{-t-1}G(q, s) \tag{15}$$

The counterclockwise contour C_0 around $s=0$ can be replaced by the sum of the clockwise contours around the poles $s_i(q)$ of $G(q, s)$. The dominant large-t contributions come from regions around saddle points q_0, where $\ln|s_i(q)| \propto |q-q_0|^2$. In cases with cubic symmetry there are two saddle points,(4) one located at the origin, $q_0=(0, 0,..., 0)$, and the other at the corner of the first Brillouin zone, $q_0=(\pi, \pi,..., \pi)$. The contribution from the saddle point at $(\pi, \pi,..., \pi)$ is equal to the contribution from the saddle point at $(0, 0,..., 0)$ multiplied by a factor $\exp[i\pi(t+\sum_\alpha n_\alpha)]$, as can be seen by changing integration variables $q'_\alpha=\pi-q_\alpha$ and using the symmetry $G(\pi-q_\alpha, s)=G(q_\alpha, -s)$. Thus, it follows that $P(n, t)$ vanishes identically if $(t+\sum_\alpha n_\alpha)$ is odd, and is twice the contribution from the saddle point at $(0, 0,..., 0)$ for $(t+\sum_\alpha n_\alpha)$ even. Thus, we consider only the latter saddle point.

For the unrestricted RW the generating function $G(q, s)= [1-s\phi(q)]^{-1}$ has only a single pole at $s_0=1/\phi$, the contour integral yields $[\phi(q)]^t$, and the saddle point method can be applied at large t.(7) In the backward jump model, $G(q, s)$ in Eq. (8) has two poles, one of which satisfies $\ln|s_0(q)| \propto q^2$. The contour integral can be calculated and the saddle point method applied in a straightforward manner.(4)

An equivalent procedure that will be used for the general case (6) is the following. Only the pole, say $s_0(q)$, with the saddle point at $q=0$ is relevant. It is thus sufficient to represent $G(q, s)$ in the vicinity of the pole as $G(q, s)=R_0(q)/[s-s_0(q)]$. To calculate the leading behavior of $P(n, t)$, one only needs the value of the residue at $q=0$ and the location of the pole accurate to terms of order q^2 included. The resulting asymptotic form is the Gaussian

$$P(n, t) = (4\pi Dt)^{-d/2} \exp(-n^2/4Dt)$$

To find the subleading corrections proportional to $1/t$, one has to calculate the residue up to $O(q^2)$ terms included and the location of the pole up to $O(q^4)$ included.

In the general case, I follow the method sketched above. The generating function in (6) has $2d$ poles. Denoting the relevant pole again by $s_0(q)$, I obtain from (6) for residue and location

$$\begin{aligned} R_0(q) &= s_0(q)[1 + (q^2/d)a/(1-a)^2] \\ s_0(q) &= 1 + Dq^2 + \Delta q^4 \end{aligned} \tag{16}$$

where D is the diffusion coefficient (14) and

$$\begin{aligned} \Delta q^4 = {} & (Dq^4/2d)\{(1+a)/(1-a)^2 + (a+b)/[(1-a)(1-b)]\} \\ & - \sum_\alpha (Dq_\alpha^4/12)\{1 + 6(a+b)/[(1-a)(1-b)]\} \end{aligned}$$

After some algebra, I finally obtain the asymptotic expansion for the probability distribution (for $t + \sum_\alpha n_\alpha$ even):

$$\begin{aligned} P(n, t) = {} & 2\frac{\exp(-n^2/4Dt)}{(4\pi\, Dt)^{d/2}} \\ & \times \left\{1 - \left[\left(\frac{1+a}{1-a}\right)^2 + \frac{2(a+b)}{(1-a)(1-b)}\right]\frac{n^4}{64dD^3t^3}\right. \\ & + \left[1 + \frac{6(a+b)}{(1-a)(1-b)}\right]\sum_\alpha \frac{n_\alpha^4}{192D^3t^3} \\ & + \left[\frac{1+2da+a^2}{(1-a)^2} - \frac{2(d-1)(a+b)}{(1-a)(1-b)}\right]\frac{n^2}{8dD^2t^2} \\ & \left. - \left[\frac{1+2(d-1)a+a^2}{(1-a)^2} + \frac{2(d-1)(a+b)}{(1-a)(1-b)}\right]\frac{1}{8Dt} + O(t^{-2})\right\} \end{aligned} \tag{17}$$

This expression gives the asymptotic form of the distribution function for the end-to-end distance n after a large number of steps t. It reduces to Domb and Fisher's result for the backward jump model if $a = -b = \delta$.

For the forward jump model ($a = b = \varepsilon$), Claes and Van den Broeck study the approach of $P(n, t)$ to the Gaussian limit by only considering the approach of the fourth and sixth moments to this limit. Here the approach of the full distribution to this limit is given by restricting Eq. (17) to the forward jump model and by replacing the coordination number $2d$ by N in the notation of ref. 5 and the step probabilities $\alpha = \gamma + \varepsilon$ by p and $\beta = \gamma$ by $(1-p)/(N-1)$.

Another moment of interest in polymer statistics is the inverse hydrodynamic radius of gyration $\langle |n|^{-1} \rangle$, which approaches its Gaussian limiting form $(6/\pi\langle n^2 \rangle)^{-1/2} = (\pi\, Dt)^{-1/2}$. This approach has been studied in ref. 5 from an expansion of $P(n, t)$ in Hermite polynomials that is slowly convergent. Here, $\langle |n|^{-1} \rangle$ is obtained from Eq. (17) in a simple manner by integrating n over three-dimensional Euclidean space, with the result at large t

$$\langle |n|^{-1} \rangle = (\pi\, Dt)^{-1/2} (1 + A/t + \cdots)$$

with an explicit expression for A.

7. EIGENMODES OF THE CHAPMAN–KOLMOGOROV EQUATION

One may want to study the relaxational modes of the Chapman–Kolmogorov equation, defined as

$$f_\nu^{(\lambda)}(n, t+1) = s_\lambda f_\nu^{(\lambda)}(n, t) = s_\lambda^{t+1} f_\nu^{(\lambda)}(n)$$

Because of translational invariance, the eigenmodes have the form $f_\nu^{(\lambda)}(n) = f_\nu^{(\lambda)}(q)\, e^{iqn}$ and the eigenvalue equation is

$$s_\lambda(q) f_\nu = \sum_\mu e^{-iq\nu} W_{\nu\mu} f_\mu \tag{18}$$

where the transition $2d \times 2d$ matrix $W_{\nu\mu}$ has been defined below Eq. (13). At $q = 0$ it has only three different eigenvalues because of cubic symmetry. They are given together with the corresponding $2d$ eigenvectors:

$$\begin{aligned}
&s_0(0) = 1; \qquad && f_\nu = 1 \\
&s_1(0) = a; && f_\nu = \delta_{\nu,\alpha} - \delta_{\nu,-\alpha} \quad (\alpha = x, y, \ldots, d) \\
&s_2(0) = b; && f_\nu = d(\delta_{\nu\alpha} + \delta_{\nu,-\alpha}) - 1 \quad (a = x, y, \ldots, d-1)
\end{aligned} \tag{19}$$

For nonvanishing q the eigenvalues are given by the poles of $G(q, s)$ calculated in Eqs. (6), (8), and (9). In the saddle point method of Section 6 the eigenvalue $s_0(q)$ of the diffusive mode was explicitly calculated in Eq. (16) for small q values.

8. CONCLUDING REMARKS

(i) Continuous-time random walks with a Poissonian waiting time distribution are described by the master equation. For the restricted RWs of Eq. (1), the continuous-time versions are obtained by replacing $P_\nu(n, t+1)$ in Eq. (1) by $p_\nu(n, t) + \partial p_\nu(n, t)/\partial t$. The exact solution $P_\nu(n, t)$ of the discrete-time RW in Eq. (1) and its continuous-time analog are related by[3]

$$p_\nu(n, t) = e^{-t} \sum_{\tau=0}^{\infty} P_\nu(n, \tau) t^\tau/\tau!$$

The dominant long-time behavior of the probability distribution and its moments is the same for the discrete- and continuous-time cases.

(ii) The generating function for walks commencing and/or finishing in a specified direction can also be calculated. In fact, in solving Eq. (5), I obtained as an intermediate result an expression for the generating function $G_\nu(q, s)$ of walks finishing with "velocity" ν. It was expressed in terms of $G(q, s)$, calculated in Eq. (6). To obtain the generating function for walks commencing in a specified direction, one needs the backward CK equation[1] instead of the forward one, as given in Eq. (1).

(iii) The restricted RWs can be extended slightly to include states of rest.[7] Let the RW have probabilities σ and μ to go, respectively, from a moving state to a state of rest and vice versa, and a probability σ' to remain in a state of rest, with normalization

$$\sigma + \alpha + \beta + 2(d-1)\gamma = 1, \qquad \sigma' + 2d\mu = 1$$

Then, the transition matrix $W_{\nu\mu}$ is a $(2d+1)$-dimensional matrix where the labels include the state of zero velocity as $\nu, \mu = 0$.

The stationary solution of Eq. (1) for the RW without a rest state is the normalized eigenvector in Eq. (19) with eigenvalue $s_0(0) = 1$, viz. $P_\nu(n, \infty) = 1/2d$. For the RW with a rest state, $P_\nu(n, \infty) = p\delta_{\nu 0} + q(1 - \delta_{\nu 0})$. This is the right eigenvector with unit eigenvalue of the nonsymmetric matrix $W_{\nu\mu}$. In the present case $W_{\nu\mu}$ has four distinct eigenvalues, one of which is d-fold and one of which is $(d-1)$-fold degenerate due to the cubic symmetry.

The corresponding Champan–Kolmogorov equation has to be solved with initial condition $P_\nu(n, 0) = \delta_{n0} P_\nu(\infty)$. For the general case there is no simple solution, but a few special cases, such as nonvanishing μ, σ, and σ' and $\alpha = \beta = \gamma$, can be reduced to two coupled equations as in Eq. (5) and can be solved similarly. More general choices of the model parameters make the equations quickly untractable using the present method.

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REFERENCES

1. N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
2. J. W. Haus and K. W. Kehr, *Phys. Rep.* **50**:263 (1987), Section 4.
3. P. M. Binder, *Complex Systems* **1**:559 (1987); M. H. Ernst and P. M. Binder, *J. Stat. Phys.* (1988).
4. C. Domb and M. E. Fisher, *Proc. Comb. Phil. Soc.* **54**:48 (1958).
5. I. Claes and C. Van den Broeck, *J. Stat. Phys.* **49**:383 (1987).
6. E. H. Hauge, *Phys. Fluids* **13**:1201 (1970).
7. Y. Okamura, E. Blaisten-Barojas, and S. Fujita, *Phys. Rev. B* **22**:1638 (1980).